Journal of Approximation Theory **100**, 266–283 (1999)
Article ID jath.1999.3331, available online at http://www.idealibrary.com on IDEAL®

Best Harmonic and Superharmonic L^1-Approximants in Strips

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Communicated by Manfred v. Golitschek

Received September 15, 1997; accepted November 5, 1998

Let Ω denote the open strip $(-1, 1) \times \mathbb{R}^{n-1}$, where $n \geqslant 2$. We completely solve the problem of characterizing a best harmonic L^1-approximant to a subharmonic function s on Ω (all functions are assumed to be continuous and integrable on $\bar{\Omega}$). This characterization was previously known only under highly restrictive hypotheses on s. The approach of this paper is based, in part, on ideas used recently to solve the corresponding problem for the unit ball. However, the unboundedness of Ω presents difficulties which require the use of new techniques and recent results from other branches of harmonic approximation theory. Superharmonic L^1-approximation of subharmonic functions is also treated. © 1999 Academic Press

1. INTRODUCTION AND RESULTS

For certain special domains ω in $\mathbb{R}^n$ ($n \geqslant 2$) several authors (see [2, 3, 5, 9, 10]) have sought to characterize best harmonic approximants, in the L^1-norm, to subharmonic functions on ω (all functions are assumed to be continuous on $\bar{\omega}$). However, it was only very recently [2] that a complete characterization was found, even in the simplest case where ω is the unit ball. The purpose of this paper is to obtain results analogous to those in [2] for an n-dimensional strip. It turns out that, because we are now working with an unbounded domain, implementation of the strategy in [2] requires much more powerful techniques; in particular, we need to use recent results from other branches of harmonic approximation theory.

0021-9045/99 $30.00

To proceed further, we introduce some notation. Let $\mathscr{H}(\omega)$, $\mathscr{S}(\omega)$, and $\mathscr{U}(\omega)$ denote respectively the collections of harmonic, subharmonic, and superharmonic functions on ω. If $f \in L^1(\omega)$, then we define $\|f\|_1 = \int_\omega |f|$, where the integral is with respect to n-dimensional Lebesgue measure λ. A function $h^* \in C(\bar{\omega}) \cap \mathscr{H}(\omega)$ which satisfies

$$\|f - h^*\|_1 \leqslant \|f - h\|_1 \qquad \text{for all} \quad h \in C(\bar{\omega}) \cap \mathscr{H}(\omega)$$

is called a *best harmonic L^1-approximant to f on $\bar{\omega}$*. Similarly, a function $u^* \in C(\bar{\omega}) \cap \mathscr{U}(\omega)$ which satisfies

$$\|f - u^*\|_1 \leqslant \|f - u\|_1 \qquad \text{for all} \quad u \in C(\bar{\omega}) \cap \mathscr{U}(\omega)$$

is called a *best superharmonic L^1-approximant to f on $\bar{\omega}$*. Let B denote the open unit ball in $\mathbb{R}^n$ and let B_0 be the open ball of centre 0 and radius $2^{-1/n}$. Thus $\lambda(B_0) = \lambda(B)/2$. The main result of [2] is as follows.

THEOREM A. *Let $s \in C(\bar{B}) \cap \mathscr{S}(B)$ and $h^* \in C(\bar{B}) \cap \mathscr{H}(B)$. Then h^* is a best harmonic L^1-approximant to s on $\bar{B}$ if and only if*

(i) *$h^* = s$ on ∂B_0, and*

(ii) *$h^* \leqslant s$ on $\bar{B} \backslash B_0$.*

It was also shown in [2] that a best superharmonic L^1-approximant to a given subharmonic function is necessarily harmonic. Below we will establish analogues of these results for an n-dimensional strip.

Let $\Omega(k) = (-k, k) \times \mathbb{R}^{n-1}$ for each positive number k; let $\Omega = \Omega(1)$ and $\Omega_0 = \Omega(1/2)$. It will also be convenient to write

$$\mathscr{H} = C(\bar{\Omega}) \cap L^1(\Omega) \cap \mathscr{H}(\Omega), \qquad \mathscr{S} = C(\bar{\Omega}) \cap L^1(\Omega) \cap \mathscr{S}(\Omega),$$

$$\mathscr{U} = C(\bar{\Omega}) \cap L^1(\Omega) \cap \mathscr{U}(\Omega).$$

THEOREM 1. *Let $s \in \mathscr{S}$ and $h^* \in \mathscr{H}$. Then h^* is a best harmonic L^1-approximant to s on $\bar{\Omega}$ if and only if*

(i) *$h^* = s$ on $\partial \Omega_0$, and*

(ii) *$h^* \leqslant s$ on $\bar{\Omega} \backslash \Omega_0$.*

We note that Theorem 1 was proved in [5] under the additional and very strong hypotheses that $s \in C^2(\Omega)$ and that $\Delta s > 0$ almost everywhere. The corollaries below follow easily from Theorem 1.

COROLLARY 1. *Let $s \in \mathscr{S}$. If s has a best harmonic L^1-approximant h^* on $\bar{\Omega}$, then h^* is unique, and $s \leqslant h^*$ on Ω_0.*

COROLLARY 2. *If $s \in \mathscr{S}$ and the best harmonic L^1-approximant h^* to s on $\bar{\Omega}$ exists, then*

$$\|s-h^*\|_1=\int_{\Omega} s-2\int_{\Omega_0} s.$$

COROLLARY 3. *Let s_j belong to $\mathscr{S}$ $(j=1,2)$ and let h_j^* be the best harmonic L^1-approximant to s_j on $\bar{\Omega}$. Then*

(i) *$h_1^*+h_2^*$ is the best harmonic L^1-approximant to s_1+s_2 on $\bar{\Omega}$, and*

(ii) $\|s_1-h_1^*\|_1 \leqslant \|s_1+s_2-(h_1^*+h_2^*)\|_1$.

THEOREM 2. *Let $s \in \mathscr{S}$ and $u^* \in \mathscr{U}$. Then u^* is a best superharmonic L^1-approximant to s on $\bar{\Omega}$ if and only if u^* is the best harmonic L^1-approximant to s on $\bar{\Omega}$.*

The paper is organized as follows. The central part of the proof of Theorem 1 is contained in a proposition which we state and prove in Section 2. Theorem 1 and its corollaries are then deduced in Section 3, and Theorem 2 is proved in Section 4.

2. A KEY RESULT

2.1. For any function $f: \bar{\Omega} \to \mathbb{R}$ we define

$$E_+(f)=\{x \in \bar{\Omega} : f(x)>0\},$$

$$E_-(f)=\{x \in \bar{\Omega} : f(x)<0\},$$

$$E_0(f)=\{x \in \bar{\Omega} : f(x)=0\}.$$

When there is no risk of ambiguity, we write E_+ for $E_+(f)$, etc. The purpose of Section 2 is to prove the following proposition which forms the core of the proof of Theorem 1.

PROPOSITION. *Let $s \in \mathscr{S}$ and suppose that 0 is a best harmonic L^1-approximant to s on $\bar{\Omega}$. The following are equivalent:*

(a) $s \geqslant 0$ *on* $\bar{\Omega}$,

(b) $\bar{\Omega}_0 \subseteq E_0(s)$.

2.2. We begin by assembling some basic material. If $f: \overline{\Omega} \to \mathbb{R}$, then we write $f^+ = \max\{f, 0\}$. If f is integrable with respect to $(n-1)$-dimensional Lebesgue measure λ' on a hyperplane $\{t\} \times \mathbb{R}^{n-1}$, then we define

$$\mathscr{M}(f, t) = \int_{\mathbb{R}^{n-1}} f(t, x')\, d\lambda'(x').$$

It is well known (see, for example, [1]) that if $h \in \mathscr{H}$, then h is λ'-integrable on $\{t\} \times \mathbb{R}^{n-1}$ for each $t \in (-1, 1)$ and $\mathscr{M}(h, \cdot)$ is an affine function on $(-1, 1)$. From this it follows immediately that

$$\int_{\Omega(k)} h = 2k\mathscr{M}(h, 0) \qquad (h \in \mathscr{H}; 0 < k \leqslant 1). \tag{2.1}$$

This analogue of the standard mean value property of harmonic functions on a ball will be used repeatedly, as will the following lemma.

LEMMA 1. *Let $s \in \mathscr{S}$ and $h^* \in \mathscr{H}$. The following are equivalent*:

(a) *h^* is a best harmonic L^1-approximant to s on $\overline{\Omega}$*;

(b) *for every $h \in \mathscr{H}$*,

$$\int_{E_-(s-h^*)} h - \int_{E_+(s-h^*)} h + \int_{E_0(s-h^*)} |h| \geqslant 0; \tag{2.2}$$

(c) *for every $h \in \mathscr{H}$*,

$$\int_{E_0(s-h^*)} h^+ + \int_{E_-(s-h^*)} h \geqslant \mathscr{M}(h, 0). \tag{2.3}$$

The equivalence of (a) and (b) is a special case of [12, Theorem 4.5.3] (or see [14, Theorem 1.7]). The equivalence of (b) and (c) is proved as follows. Adding $2\mathscr{M}(h, 0)$ to each side of (2.2) and using (2.1) with $k = 1$, we find that (2.2) is equivalent to

$$2\mathscr{M}(h, 0) \leqslant \int_{E_0} |h| + \int_{E_0} h + 2\int_{E_-} h = 2\int_{E_0} h^+ + 2\int_{E_-} h,$$

which is equivalent to (2.3).

Next we give a simple maximum principle. We claim no originality for it, but it is easier to give a proof than an exact reference.

LEMMA 2. *Let ω be an open subset of Ω. If $s \in C(\bar{\omega}) \cap L^1(\omega) \cap \mathscr{S}(\omega)$ and $s \leqslant 0$ on $\partial\omega$, then $s \leqslant 0$ on ω. Further, if ω is connected, then either $s < 0$ on ω or $s = 0$ on ω.*

To see this, define S to be equal to s^+ on ω and 0 on $\mathbb{R}^n \backslash \omega$. Then $S \in \mathscr{S}(\mathbb{R}^n)$, so

$$S(x) \leqslant \frac{1}{\lambda(B(x, r))} \int_{B(x, r)} S \leqslant \frac{1}{\lambda(B(x, r))} \int_{\omega} s^+ \to 0 \qquad (r \to +\infty),$$

where $B(x, r)$ denotes the open ball of centre x and radius r in $\mathbb{R}^n$. The stronger conclusion when ω is connected follows from the classical maximum principle.

2.3. The lemma below will be used to prove that (a) implies (b) in the proposition.

LEMMA 3. *Let F be a proper closed subset of $\bar{\Omega}$ such that each component of $\mathbb{R}^n \backslash F$ meets $\mathbb{R}^n \backslash \bar{\Omega}$. If $y_0 \in \Omega \cap \partial F$, $a > 1$, and $h \in L^1(\Omega(a)) \cap \mathscr{H}(\Omega(a) \backslash \{y_0\})$, then for every $\varepsilon > 0$ there exists $H \in \mathscr{H}(\mathbb{R}^n)$ such that*

$$\int_F |h - H| < \varepsilon. \tag{2.4}$$

For each $y \in \mathbb{R}^n$ let $h_y(x) = h(x + y)$. If $\|y\| < a - 1$, then $h_y \in \mathscr{H}(\Omega \backslash \{y_0 - y\})$. We first claim that

$$\int_{\bar{\Omega}} |h - h_y| \to 0 \qquad (y \to 0). \tag{2.5}$$

To see this, let $\eta > 0$. Since $h \in L^1(\Omega(a))$, there exist positive numbers r and R such that $B(y_0, r) \subseteq \Omega \cap B(0, R)$ and

$$\int_{B(y_0, r)} |h_y| < \eta/6, \qquad \int_{\bar{\Omega} \backslash \overline{B(0, R)}} |h_y| < \eta/6 \tag{2.6}$$

whenever $\|y\|$ is sufficiently small. On the compact set $(\bar{\Omega} \cap \overline{B(0, R)}) \backslash B(y_0, r) = T$, say, the functions h_y are uniformly bounded for small values of $\|y\|$, and they converge pointwise to h as $\|y\| \to 0$. Hence, if $\|y\|$ is sufficiently small,

$$\int_T |h - h_y| < \eta/3. \tag{2.7}$$

If both (2.6) and (2.7) hold, then

$$\int_{\bar{\Omega}} |h - h_y| < \eta.$$

This establishes (2.5).

Since $y_0 \in \Omega \cap \partial F$, it follows from (2.5) that there exists $y_1 \in \Omega \backslash F$ such that the function $H_0 = h_{y_0 - y_1}$ is harmonic on $\Omega(b) \backslash \{y_1\}$, where $b = (a+1)/2$, and satisfies

$$\int_{\bar{\Omega}} |h - H_0| < \varepsilon/2. \tag{2.8}$$

Let ω_0 be the component of $\mathbb{R}^n \backslash F$ which contains y_1. By hypothesis, ω_0 meets $\mathbb{R}^n \backslash \overline{\Omega(b)}$. Let ω_1 be a bounded open connected set such that $y_1 \in \omega_1$, $\bar{\omega}_1 \subset \omega_0$, and $\omega_1 \backslash \overline{\Omega(b)} \neq \varnothing$, and define $E = \bar{\Omega} \backslash \omega_1$. Then E is closed and H_0 is harmonic on a neighbourhood of E. Also, the complement of E in the one-point compactification of $\mathbb{R}^n$ is connected and locally connected. It now follows from [4, Theorem 1.1] (or see [8, Corollary 5.10]) that if d is a positive constant, then there exists $H \in \mathscr{H}(\mathbb{R}^n)$ such that

$$|H(x) - H_0(x)| < d(1 + \|x\|)^{-n-1} \qquad (x \in E).$$

Since $H_0 \in L^1(\Omega(a))$, we have $H \in L^1(\Omega(a))$. By choosing d small enough, we can therefore arrange that

$$\int_E |H - H_0| < \varepsilon/2. \tag{2.9}$$

Since $F \subseteq E \subseteq \bar{\Omega}$, (2.4) follows from (2.8) and (2.9).

2.4. We can now prove that (a) implies (b) in the proposition. Suppose that 0 is a best harmonic L^1-approximant to s and that $s \geqslant 0$ on $\bar{\Omega}$. We write $E_0 = E_0(s)$ and define a number τ_1 as follows: if E_0 contains no strip $\Omega(t)$, then $\tau_1 = 0$; otherwise define $\tau_1 = \sup\{t \colon \Omega(t) \subseteq E_0\}$. We have to show that $\tau_1 \geqslant 1/2$. Suppose, to the contrary, that $\tau_1 < 1/2$, and let τ_2 be such that $\max\{\tau_1, 1/3\} < \tau_2 < 1/2$. Define F_0 to be the union of the set $E_0 \cup \overline{\Omega(\tau_2)}$ with all the components of $\mathbb{R}^n \backslash (E_0 \cup \overline{\Omega(\tau_2)})$ that are contained in Ω. We claim that

$$\partial F_0 \cap (\partial \Omega(\tau_2) \backslash E_0) \neq \varnothing. \tag{2.10}$$

To verify (2.10), observe first that

$$\partial F_0 \subseteq \partial(E_0 \cup \overline{\Omega(\tau_2)}) \subseteq \partial E_0 \cup \partial \Omega(\tau_2).$$

Hence, if (2.10) is false, $\partial F_0 \subseteq \partial E_0$ and therefore $s=0$ on ∂F_0, so that by Lemma 2, $s=0$ on $(F_0)^\circ$. We now have $s=0$ on F_0, so that $\Omega(\tau_2) \subseteq F_0 \subseteq E_0$ and $\tau_2 \leqslant \tau_1$, contrary to our choice of τ_2. This establishes (2.10).

Now choose a point $y_0 \in \partial F_0 \cap (\partial\Omega(\tau_2) \backslash E_0)$. Define P to be the Poisson kernel of $\Omega(\tau_2)$, with pole at y_0, normalized so that $\mathscr{M}(P, 0)=1$, and extended by repeated reflection to be harmonic on $\mathbb{R}^n$, except for a sequence of singularities. (For properties of P we refer to [6]. The extension is possible since $P(x) \to 0$ as $x \to y$ for each $y \in \partial\Omega(\tau_2) \backslash \{y_0\}$.) Also, since $\tau_2 > 1/3$, none of the singularities of P, except y_0, lies in $\bar{\Omega}$. We note that

$$P \in L^1(\Omega(a)) \cap \mathscr{H}(\Omega(a) \backslash \{y_0\}) \qquad \text{for some} \quad a > 1,$$

$$P > 0 \qquad \text{on } \Omega(\tau_2),$$

$$P < 0 \qquad \text{on } \bar{\Omega} \backslash \overline{\Omega(\tau_2)}.$$

Let $0 < \varepsilon < (1 - 2\tau_2)/4$. Then $\varepsilon < 1/12 < \tau_2$. Since F_0 is a closed subset of $\bar{\Omega}_0$, and since $\mathbb{R}^n \backslash F_0$ consists of those components of $\mathbb{R}^n \backslash (E_0 \cup \overline{\Omega(\tau_2)})$ that meet $\mathbb{R}^n \backslash \bar{\Omega}$, we can apply Lemma 3 to obtain $H \in \mathscr{H}$ such that

$$\int_{F_0} |P - H| < \varepsilon. \tag{2.11}$$

Since $\Omega(\tau_2) \subseteq F_0$, we have (see (2.1))

$$|\mathscr{M}(P, 0) - \mathscr{M}(H, 0)| \leqslant \frac{1}{2\tau_2} \int_{\Omega(\tau_2)} |P - H| < \frac{\varepsilon}{2\tau_2} < \frac{1}{2}. \tag{2.12}$$

Since $\mathscr{M}(P, 0) = 1$, it follows that

$$\mathscr{M}(H, 0) > \tfrac{1}{2}. \tag{2.13}$$

On $F_0 \backslash \overline{\Omega(\tau_2)}$ we have $H < H - P$ and hence $H^+ \leqslant (H-P)^+ \leqslant |H-P|$. Also, $H^+ \leqslant |H| \leqslant |H-P| + P$ on $\Omega(\tau_2)$. Since $E_0 \cup \Omega(\tau_2) \subseteq F_0$, we obtain

$$\begin{aligned}\int_{E_0} H^+ &\leqslant \int_{F_0 \backslash \Omega(\tau_2)} |H-P| + \int_{\Omega(\tau_2)} |H-P| + \int_{\Omega(\tau_2)} P \\ &= \int_{F_0} |H-P| + \int_{\Omega(\tau_2)} P \\ &< \varepsilon + 2\tau_2 \mathscr{M}(P, 0),\end{aligned}$$

by (2.11) and (2.1). Using (2.12), our choice of ε, and (2.13), we obtain

$$\begin{aligned}\int_{E_0} H^+ &< \varepsilon + 2\tau_2(|\mathscr{M}(P,0) - \mathscr{M}(H,0)| + \mathscr{M}(H,0)) \\ &< 2\varepsilon + 2\tau_2 \mathscr{M}(H,0) \\ &< (1-2\tau_2)(\tfrac{1}{2} - \mathscr{M}(H,0)) + \mathscr{M}(H,0) \\ &< \mathscr{M}(H,0).\end{aligned}$$

This contradicts (2.3). Hence $\tau_1 \geqslant 1/2$, as required.

2.5. In order to prove the converse implication in the proposition we need the following lemma.

LEMMA 4. *Let F be a proper closed subset of $\mathbb{R}^n$, let $1 < p \leqslant +\infty$, and suppose that $h \in L^p(F) \cap \mathscr{H}(F^o)$. Then, for each $\varepsilon > 0$, there exists a function H, harmonic on $\mathbb{R}^n$ except for isolated singularities in $\mathbb{R}^n \backslash F$, such that*

$$\int_F |h - H| < \varepsilon.$$

To prove this, let $1 < q < \min\{p, n/(n-1)\}$ and define $F_k = F \cap \overline{B(0,k)}$ for $k = 1, 2, ...$ By a theorem of Hedberg [11] (see also [13]), for each k there is a harmonic function h_k on a neighbourhood of F_k such that

$$\int_{F_k} |h_k - h| \leqslant \left(\int_{F_k} |h_k - h|^q \right)^{1/q} (\lambda(F_k))^{(q-1)/q} < \varepsilon.$$

It now follows from [4, Theorem 1.4] (or rather, from its proof) that there is a function H with the stated properties.

2.6. We now prove that (b) implies (a) in the proposition. Suppose that 0 is a best harmonic L^1-approximant to s and that $\bar{\Omega}_0 \subseteq E_0(s)$. We write $E_+ = E_+(s)$, etc. Observe first that

$$(E_0 \cup E_-)^\circ = E_0^\circ \cup E_-, \tag{2.14}$$

for if there were a point y_0 of ∂E_0 in $(E_0 \cup E_-)^\circ$, then the mean value inequality for the subharmonic function s would fail for small balls centred at y_0.

We now suppose that $E \neq \varnothing$ and show that this leads to a contradiction. Let $\partial^\infty E_-$ denote the boundary of E_- in the one-point compactification $\mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$ and let μ_x be harmonic measure on $\partial^\infty E_-$ relative to a point x of E_-. Note that, if $\infty \in \partial^\infty E_-$, then $\mu_x(\{\infty\}) = 0$ for each $x \in E_-$.

To see this, take a positive harmonic function g on Ω such that $g(y) \to +\infty$ as $y \to \infty$, and observe that $\mu_x(\{\infty\}) < \varepsilon g(x)$ for all $\varepsilon > 0$. (An example of such a function g is given by

$$g(x_1, ..., x_n) = \cos x_1 \cosh \frac{x_2}{\sqrt{(n-1)}} \cdots \cosh \frac{x_n}{\sqrt{(n-1)}}.)$$

It follows that $\mu_x(\partial E_-) = 1$ for each $x \in E_-$.

Let $K(r)$ be the compact cylinder given by

$$K(r) = \{(t, x') \in [-1, 1] \times \mathbb{R}^{n-1} : \|x'\| \leqslant r\} \qquad (r > 0).$$

Since $\mu_x(\partial E_-) > 0$ for each $x \in E_-$, there is a number r_0 such that $\mu_x(K(r_0) \cap \partial E_-) > 0$ for some $x \in E_-$. Define $w(x) = \mu_x(K(r_0) \cap \partial E_-)$ for each $x \in E_-$ and define $w = 0$ on $\Omega \backslash E_-$. Then w is harmonic on E_- and $w(x) \to 0$ as $x \to y$ for every $y \in \partial E_- \backslash K(r_0)$ that is regular for the Dirichlet problem on E_-. Since the set of all irregular boundary points is polar, it follows that the function w^*, defined by

$$w^*(y) = \limsup_{x \to y} w(x) \qquad (y \in \Omega),$$

is subharmonic on $\Omega \backslash (\partial E_- \cap K(r_0))$ (see, for example, [7, p. 60]). Clearly

$$0 \leqslant w^* \leqslant 1 \qquad \text{on } \Omega \tag{2.15}$$

and

$$\lim_{x \to y} w^*(x) = 0 \qquad (y \in \partial^\infty \Omega \backslash K(r_0)). \tag{2.16}$$

We define

$$F_r = E_0 \cup E_- \cup (\bar{\Omega} \backslash (K(r))^\circ) \qquad (r > r_0),$$

and note from (2.14) that $w^* \in \mathcal{S}((F_r)^\circ)$. Let h_r denote the least harmonic majorant of w^* on $(F_r)^\circ$, and let $h_r = 0$ on $\bar{\Omega} \backslash (F_r)^\circ$. We need to establish the following:

(α) $h_r \in L^p(\bar{\Omega})$ for each $p \geqslant 1$;

(β) $h_r(x)$ is a decreasing function of r for each $x \in (E_0 \cup E_-)^\circ$;

(γ) $h_r(x) \to 0$ as $r \to +\infty$ for each $x \in (E_0)^\circ$.

To prove (α), let G denote the Green function for $\Omega(2)$ with pole 0, choose a constant c such that $cG \geqslant 1$ on $K(r_0)$, and define $U = \min\{1, cG\}$

on $\bar{\Omega}$. Then $U \in C(\bar{\Omega}) \cap \mathscr{U}(\Omega)$ and $U > 0$ on Ω. On $K(r_0)$ we have $0 \leqslant w^* \leqslant 1 = U$. From (2.15), (2.16), and the maximum principle, it follows that $w^* \leqslant U$ on $\Omega \backslash K(r_0)$. Hence U is a superharmonic majorant of w^* on Ω, which contains $(F_r)^\circ$, and so $0 \leqslant w^* \leqslant h_r \leqslant U$ on $(F_r)^\circ$. Since $G(x)$ decays exponentially as $x \to \infty$ (see [6]), (α) now follows.

To prove (β) we simply note that F_r decreases as r increases.

To prove (γ), we define u_r on $(F_r)^\circ$ by

$$u_r = \inf\{u \in \mathscr{U}((F_r)^\circ) : u > 0 \text{ on } (F_r)^\circ \text{ and } u \geqslant 1 \text{ on } E_- \cup (\Omega \backslash K(r))\}$$

and let $\hat{u}_r$ be the balayage given by

$$\hat{u}_r(y) = \min\{u_r(y), \liminf_{x \to y} u_r(x)\} \qquad (y \in (F_r)^\circ).$$

Then $\hat{u}_r$ is a superharmonic majorant of w^* on $(F_r)^\circ$, and so $h_r \leqslant u_r$ on $(E_0 \cap K(r))^\circ$. Hence, in view of (2.14), if $x \in (E_0)^\circ$ and $\|x\| < r$, we have $h_r(x) \leqslant v_r(x)$, where $v_r(y)$ is the harmonic measure of the set $\overline{E_0 \cap \partial K(r) \cap \Omega}$ at a point $y \in (E_0 \cap K(r))^\circ$. Since $v_r(x) \to 0$ as $r \to +\infty$ for each $x \in (E_0)^\circ$, the claim (γ) is proved.

From (α), (β), (γ), and dominated convergence, it now follows that

$$\int_{E_0} h_r \to 0 \qquad (r \to +\infty). \tag{2.17}$$

Let

$$\varepsilon = \tfrac{1}{8} \int_{E_-} w. \tag{2.18}$$

Since w is non-negative, harmonic, and not identically 0 on E_-, we see that $\varepsilon > 0$. By (2.17) we can choose $r_1 > r_0$ such that

$$\int_{E_0} h_{r_1} < \varepsilon. \tag{2.19}$$

Define $h_0 = -h_{r_1}$ and $F = F_{r_1}$. By (α), $h_0 \in L^p(F)$ for each $p \geqslant 1$, and $h_0 \in \mathscr{H}(F^\circ)$ by definition. Hence, by Lemma 4, there exists a function H, harmonic on $\mathbb{R}^n$ apart from isolated singularities in $\mathbb{R}^n \backslash F$, such that

$$\int_F |h_0 - H| < \varepsilon. \tag{2.20}$$

Since $\bar{\Omega} \backslash (K(r_1))^\circ \subseteq F$, only finitely many singularities of H lie in $\bar{\Omega}$. Each singularity in $\bar{\Omega} \backslash F$ lies in E_+. By Lemma 2 and the continuity of s, every component of E_+ meets $\partial\Omega$. It now follows from [4, Lemma 4.1] that there exists a harmonic function h on a neighbourhood of $\bar{\Omega}$ such that

$$\int_{E_0 \cup E_-} |H-h| < \varepsilon. \tag{2.21}$$

In fact, the range of integration $E_0 \cup E_-$ in (2.21) can be replaced by $\bar{\Omega}\backslash L$, where L is some bounded subset of E_+. With this modification, it follows from (2.20), (2.21), and property (α) that $h \in L^1(\bar{\Omega})$, and so $h \in \mathscr{H}$.

Since $\Omega_0 \subseteq E_0$, we see from (2.1) and (2.19) that

$$|\mathscr{M}(h_0, 0)| = \left|\int_{\Omega_0} h_0\right| = \int_{\Omega_0} h_{r_1} < \varepsilon. \tag{2.22}$$

Since $h - h_0 \in \mathscr{H}(\Omega_0)$, it follows from (2.1), (2.20), and (2.21) that

$$|\mathscr{M}(h, 0) - \mathscr{M}(h_0, 0)| = \left|\int_{\Omega_0} h - h_0\right| < 2\varepsilon. \tag{2.23}$$

From (2.19)–(2.21),

$$\int_{E_0} h^+ \leqslant \int_{E_0} |h| \leqslant \int_{E_0} |h-h_0| + \int_{E_0} |h_0| < 3\varepsilon. \tag{2.24}$$

Finally, since $h_{r_1} \geqslant w^* = w$ on E_-, we have $h_0 \leqslant -w$ on E_- and so

$$\begin{aligned}
\int_{E_0} h^+ + \int_{E_-} h &< 3\varepsilon + \int_{E_-} (h - h_0 - w) && \text{(by (2.24))}\\
&\leqslant 3\varepsilon + \int_{E_-} |h-h_0| - \int_{E_-} w \\
&< 3\varepsilon + 2\varepsilon - 8\varepsilon && \text{(by (2.20), (2.21), (2.18))}\\
&< \mathscr{M}(h, 0) && \text{(by (2.22), (2.23)).}
\end{aligned}$$

This contradicts (2.3). Hence $E_- = \varnothing$.

3. PROOF OF THEOREM 1 AND COROLLARIES

3.1. We begin with the sufficiency of conditions (i) and (ii) in Theorem 1. Note first that, if $h \in \mathscr{H}$, then

$$\int_{\Omega} h = 2\mathscr{M}(h, 0) = 2\int_{\Omega_0} h$$

by (2.1), and so

$$\int_{\Omega\setminus\Omega_0} h = \int_{\Omega_0} h. \tag{3.1}$$

Now suppose that (i) and (ii) hold. By Lemma 2, applied to $s-h^*$, either

(I) $s-h^*=0$ on Ω_0, or

(II) $s-h^*<0$ on Ω_0.

Let $E_+ = E_+(s-h^*)$, etc. If (I) holds, then $E_+ \subseteq \bar{\Omega}\setminus\Omega_0$ and $E_- = \varnothing$ by (ii), so that for every $h\in\mathscr{H}$,

$$\begin{aligned}\int_{E_-} h - \int_{E_+} h + \int_{E_0} |h| &= -\int_{E_+} h + \int_{E_0\setminus\Omega_0} |h| + \int_{\Omega_0} |h| \\ &\geqslant -\int_{E_+} h - \int_{E_0\setminus\Omega_0} h + \int_{\Omega_0} |h| \\ &= -\int_{\bar{\Omega}\setminus\Omega_0} h + \int_{\Omega_0} |h| \geqslant 0,\end{aligned}$$

by (3.1). Hence (2.2) holds.

If (II) holds, then $E_- = \Omega_0$ by (ii). Hence, for every $h\in\mathscr{H}$,

$$-\int_{E_-} h = -\int_{E_+\cup E_0} h \leqslant -\int_{E_+} h + \int_{E_0} |h|,$$

by (3.1), and again (2.2) holds.

It now follows from Lemma 1 that h^* is a best harmonic L^1-approximant to s.

3.2. It remains to demonstrate the necessity of (i) and (ii). Let h^* be a best harmonic L^1-approximant to s. By considering $s-h^*$ instead of s, we may suppose that 0 is a best harmonic L^1-approximant to s. Clearly $E_0(s^+) = E_0(s)\cup E_-(s)$, $E_+(s^+) = E_+(s)$ and $E_-(s^+) = \varnothing$. Thus, for each $h\in\mathscr{H}$,

$$\begin{aligned}\int_{E_-(s^+)} h - \int_{E_+(s^+)} h + \int_{E_0(s^+)} |h| &= -\int_{E_+(s)} h + \int_{E_-(s)} |h| + \int_{E_0(s)} |h| \\ &\geqslant -\int_{E_+(s)} h + \int_{E_-(s)} h + \int_{E_0(s)} |h| \\ &\geqslant 0,\end{aligned}$$

by Lemma 1. Hence, again by Lemma 1, the function 0 is a best harmonic L^1-approximant to $s^+ \in \mathscr{S}$. By the implication "(a) $\Rightarrow$ (b)" in the proposition, applied to s^+,

$$\bar{\Omega}_0 \subseteq E_0(s^+) = E_0(s) \cup E_-(s).$$

In particular, $s \leqslant 0$ on $\partial\Omega_0$. Hence, by Lemma 2, either

(I) $s = 0$ on Ω_0, or

(II) $s < 0$ on Ω_0.

In case (I) we apply the implication "(b) $\Rightarrow$ (a)" in the proposition to see that $s \geqslant 0$ on $\bar{\Omega}$, so conditions (i) and (ii) hold.

To deal with case (II), choose $h \in \mathscr{H}$ so that $\mathscr{M}(h, t) = -1$ for each $t \in (-1, 1)$; for example, we can take h to be a suitable multiple of the Poisson kernel for $(-2, +\infty) \times \mathbb{R}^{n-1}$ with pole $(-2, 0, ..., 0)$:

$$h(t, x') = -ct\{\|x'\|^2 + (t+2)^2\}^{-n/2} \qquad (t > -2;\ x' \in \mathbb{R}^{n-1}).$$

Applying (2.2) with $h^* = 0$ we obtain

$$-1 = \int_{\Omega_0} h \geqslant \int_{E_-(s)} h \geqslant \int_{E_+(s) \cup E_0(s)} h \geqslant \int_{\Omega \backslash \Omega_0} h = -1,$$

so that equality holds throughout. Hence $s \geqslant 0$ almost everywhere on $\bar{\Omega} \backslash \Omega_0$, and therefore, by continuity, $s \geqslant 0$ on $\bar{\Omega} \backslash \Omega_0$. Since $s < 0$ on Ω_0, we have $s = 0$ on $\partial\Omega_0$ by continuity. Thus (i) and (ii) again hold.

3.3. Corollary 1 follows easily, since by Theorem 1 any two best harmonic L^1-approximants to $s \in \mathscr{S}$ must agree on $\partial\Omega_0$, and hence on Ω_0 by Lemma 2, and thus on all of $\bar{\Omega}$. Also, if h^* is the best harmonic L^1-approximant to s, then $s - h^* = 0$ on $\partial\Omega_0$, and hence $s - h^* \leqslant 0$ on Ω_0 by Lemma 2 again.

To prove Corollary 2, observe that, by Theorem 1 and Corollary 1, $s - h^* \leqslant 0$ on Ω_0 and $s - h^* \geqslant 0$ on $\Omega \backslash \Omega_0$, and hence

$$\|s - h^*\|_1 = \int_{\Omega_0} (h^* - s) + \int_{\Omega \backslash \Omega_0} (s - h^*),$$

so that the result follows by (3.1).

In Corollary 3 we have $h_k^* = s_k$ on $\partial\Omega_0$ and $h_k^* \leqslant s_k$ on $\bar{\Omega} \backslash \Omega_0$ $(k = 1, 2)$, so $h_1^* + h_2^* = s_1 + s_2$ on $\partial\Omega_0$ and $h_1^* + h_2^* \leqslant s_1 + s_2$ on $\bar{\Omega} \backslash \Omega_0$ and (i) follows. Further,

$$s_1 + s_2 - (h_1^* + h_2^*) \leqslant s_1 - h_1^* \leqslant 0 \qquad \text{on } \Omega_0$$

by Corollary 1, and

$$s_1 + s_2 - (h_1^* + h_2^*) \geqslant s_1 - h_1^* \geqslant 0 \qquad \text{on } \bar{\Omega} \backslash \Omega_0,$$

so (ii) also holds.

4. PROOF OF THEOREM 2.

4.1. We need the following lemmas.

LEMMA 5. *Let $s \in \mathscr{S}$ and $u^* \in \mathscr{U}$. Then u^* is a best superharmonic L^1-approximant to s on $\bar{\Omega}$ if and only if*

$$\int_{E_-(s-u^*)} (u-u^*) - \int_{E_+(s-u^*)} (u-u^*) + \int_{E_0(s-u^*)} |u-u^*| \geqslant 0 \qquad (4.1)$$

for every $u \in \mathscr{U}$.

LEMMA 6. *If $u \in \mathscr{U}$, then*

$$\int_{\bar{\Omega}\backslash\Omega_0} u \leqslant \int_{\Omega_0} u \qquad \textit{and} \qquad \int_{\bar{\Omega}} u \leqslant 2\mathscr{M}(u, 0).$$

Further, in each of these inequalities, equality holds if and only if $u \in \mathscr{H}$.

Lemma 5 is a special case of [12, Theorem 4.5.3]; it depends on the fact that $\mathscr{U}$ is a convex set.

The proof of Lemma 6 depends on the fact that if $u \in \mathscr{U}$, then $\mathscr{M}(u, \cdot)$ is concave on $(-1, 1)$, and is affine if and only if $u \in \mathscr{H}$ (see, for example, [1]). This implies that the function

$$\Phi(t) = \mathscr{M}(u, t) + \mathscr{M}(u, -t) \qquad (0 \leqslant t < 1)$$

is decreasing on $[0, 1)$, and is constant if and only if $u \in \mathscr{H}$. Hence

$$\int_{\bar{\Omega}\backslash\Omega_0} u = \int_0^{1/2} \Phi(t + \tfrac{1}{2})\, dt \leqslant \int_0^{1/2} \Phi(t)\, dt = \int_{\Omega_0} u$$

and

$$\int_{\bar{\Omega}} u = \int_0^1 \Phi(t)\, dt \leqslant \Phi(0) = 2\mathscr{M}(u, 0),$$

with equality in each case if and only if $u \in \mathscr{H}$.

4.2. We begin the proof of Theorem 2 by showing that if $s \in \mathcal{S}$ has a best harmonic L^1-approximant u^*, then u^* is also a best superharmonic L^1-approximant to s. Without loss of generality we may assume that $u^* \equiv 0$. By Theorem 1 and Lemma 2, $s \geqslant 0$ on $\bar{\Omega} \backslash \Omega_0$ and either

(I) $s = 0$ on Ω_0, or

(II) $s < 0$ on Ω_0.

It is enough in each case to show that (4.1) holds for every $u \in \mathcal{U}$.
In case (I), $E_-(s) = \varnothing$ and $\Omega_0 \subseteq E_0(s)$. Hence

$$-\int_{E_+(s)} u + \int_{E_0(s)} |u| \geqslant -\int_{\bar{\Omega} \backslash \Omega_0} u + \int_{\Omega_0} u \geqslant 0$$

by Lemma 6. Hence (4.1) holds.
In case (II), $E_- = \Omega_0$, so by Lemma 6,

$$\int_{E_-(s)} u = \int_{\Omega_0} u \geqslant \int_{\bar{\Omega} \backslash \Omega_0} u = \int_{E_+(s) \cup E_0(s)} u$$

$$\geqslant \int_{E_+(s)} u - \int_{E_0(s)} |u|,$$

so that (4.1) again holds.

4.3. Now suppose that $u^* \in \mathcal{U}$ is a best superharmonic L^1-approximant to s and that $u^* \notin \mathcal{H}$. We will show that this leads to a contradiction. Since $u^* \notin \mathcal{H}$, the Riesz measure μ associated with u^* is not the zero measure. Let K be a compact subset of Ω such that $\mu(K) > 0$, and let u_1^* be the Green potential on Ω with Riesz measure $\mu|_K$. Then $u_1^* \in C(\Omega)$ since $u^* \in C(\Omega)$, and u_1^* has a continuous extension to $\bar{\Omega}$, given by defining $u_1^* = 0$ on $\partial\Omega$. Also, since $G_\Omega(x, y)$, the Green function for Ω, decays exponentially as $\|y\| \to +\infty$, uniformly for x in K (see [6]), it follows that u_1^* also decays exponentially and therefore $u_1^* \in L^1(\Omega)$. Thus $u_1^* \in \mathcal{U}$. Now define $s_1 = s - (u^* - u_1^*)$. Then $u^* - u_1^* \in \mathcal{U}$ and $s_1 \in \mathcal{S}$, and u_1^* is a best superharmonic L^1-approximant to s_1.
It follows that 0 is a best superharmonic L^1-approximant to $s_1 - u_1^* = s - u^*$, and therefore 0 is a best harmonic L^1-approximant to $s_1 - u_1^*$. By Theorem 1 and Lemma 2, $s_1 \geqslant u_1^*$ on $\bar{\Omega} \backslash \Omega_0$ and either

(I) $s_1 - u_1^* = 0$ on Ω_0, or

(II) $s_1 - u_1^* < 0$ on Ω_0.

We write $E_+ = E_+(s_1 - u_1^*)$, etc. In case (I), $E_- = \emptyset$ and $\Omega_0 \subseteq E_0$. Since $u_1^* \notin \mathscr{H}$, it follows from Lemma 6 that

$$2\mathscr{M}(u_1^*, 0) > \int_{\bar{\Omega}} u_1^*.$$

Let

$$\varepsilon = \frac{1}{8}\left(2\mathscr{M}(u_1^*, 0) - \int_{\bar{\Omega}} u_1^*\right). \tag{4.2}$$

Since $s_1 = u_1^*$ on E_0, we have $u_1^* \in \mathscr{H}((E_0)^\circ)$. Let $K(r)$ be the compact cylinder defined in Section 2.6, and choose r so that $K \subset (K(r))^\circ$. Define $F = E_0 \cup \overline{(\Omega \backslash K(r))}$. Then $u_1^* \in L^p(F) \cap \mathscr{H}(F^\circ)$ for each $p \geqslant 1$. By Lemma 4, there is a function H, harmonic on $\mathbb{R}^n$ except for isolated singularities in $\mathbb{R}^n \backslash F$, such that

$$\int_F |H - u_1^*| < \varepsilon. \tag{4.3}$$

Since $\overline{\Omega \backslash K(r)} \subseteq F$, only finitely many singularities of H lie in $\bar{\Omega}$, and these singularities also lie in E_+. Since, by Lemma 2, each component of E_+ meets $\partial\Omega$, it follows from [4, Lemma 4.1] that there exists $h \in \mathscr{H}$ such that

$$\int_{E_0} |h - H| < \varepsilon. \tag{4.4}$$

Since $\Omega_0 \subseteq E_0$, we see from (2.1), (4.3), and (4.4) that

$$|\mathscr{M}(h, 0) - \mathscr{M}(u_1^*, 0)| = \left|\int_{\Omega_0} (h - u_1^*)\right| < 2\varepsilon. \tag{4.5}$$

Now

$$\begin{aligned} \int_{E_+} (h - u_1^*) &= \int_{\bar{\Omega}} (h - u_1^*) - \int_{E_0} (h - u_1^*) \\ &> \int_{\bar{\Omega}} (h - u_1^*) - 2\varepsilon \qquad \text{(by (4.3), (4.4))} \\ &= 2\mathscr{M}(h, 0) - \int_{\bar{\Omega}} u_1^* - 2\varepsilon \qquad \text{(by (2.1))} \end{aligned}$$

$$\geqslant 2\mathscr{M}(u_1^*, 0) - 4\varepsilon - \int_{\bar{\Omega}} u_1^* - 2\varepsilon \qquad \text{(by (4.5))}$$

$$= 2\varepsilon > \int_{E_0} |h - u_1^*|, \qquad \text{(by (4.2)–(4.4))},$$

contradicting Lemma 5.

In case (II), $E_- = \Omega_0$. Taking $u = 0$ in (4.1) we obtain

$$\int_{E_-} u_1^* \leqslant \int_{E_+ \cup E_0} u_1^* = \int_{\bar{\Omega} \setminus \Omega_0} u_1^* \leqslant \int_{E_-} u_1^*$$

(the last inequality follows from Lemma 6). Hence

$$\int_{\bar{\Omega} \setminus \Omega_0} u_1^* = \int_{\Omega_0} u_1^*,$$

and so $u_1^* \in \mathscr{H}$ by Lemma 6. This contradicts our construction of u_1^*.

We now have a contradiction in each case, so $u^* \in \mathscr{H}$ as required.

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